Published by Institute of Physics Publishing for SISSA

Received: *October 1, 2007*
Accepted: *October 25, 2007*
Published: *December 4, 2007*

Supersymmetric states in M5/M2 CFTs

Sayantani Bhattacharyya and Shiraz Minwalla

Department of Theoretical Physics,
Tata Institute of Fundamental Research,
Homi Bhabha Rd, Mumbai 400 005
E-mail: sayanta@theory.tifr.res.in, minwalla@theory.tifr.res.in

Abstract: We propose an exact, finite N formula for the partition function over $1/4^{\text{th}}$ BPS states in the conformal field theory on the world volume of N coincident $M5$ branes, and $1/8^{\text{th}}$ BPS states in the theory of N conincident $M2$ branes. We obtain our partition function by performing the radial quantization of the Coulomb Branches of these theories and rederive the same formula from the quantization of supersymmetric giant and dual giant gravitons in $AdS_7 \times S^4$ and $AdS_4 \times S^7$. Our partition function is qualitatively similar to the analogous quantity in $\mathcal{N} = 4$ Yang Mills. It reduces to the sum over supersymmetric multi gravitons at low energies, but deviates from this supergravity formula at energies that scale like a positive power of N.

Keywords: Extended Supersymmetry, M-Theory, AdS-CFT Correspondence.

JHEP12(2007)004

Contents

1. Introduction **1**

2. Supersymmetric states and the supergravity partition function **3**
- 2.1 M5 branes 3
- 2.2 M2 Branes 4

3. Quantization of the Coulomb branch **5**
- 3.1 Radial quantization of a free scalar fields in aribtrary dimension 5
- 3.2 Quantization of the BPS sector of the U(1) theory 8
- 3.3 Quantization of the Coulomb branch 9
- 3.4 Absence of additional supersymmetric configurations in $\mathcal{N} = 4$ Yang Mills 9
- 3.5 Dual giants: bulk duals of the Coulomb branch 10

4. Supersymmemtric states from giant gravitons **13**
- 4.1 Quantization of linear polynomials 14
- 4.2 Calculation of $\omega_{\rm BI}$ 16
- 4.3 Calculation of $\omega_{\rm WZ}$ 18

5. Discussion **18**

A. Charges of the giant graviton **19**

1. Introduction

The low energy dynamics on the world volume of N coincident $M5$ or $M2$ branes is governed by maximally supersymmetric conformal field theories in $d = 6$ and $d = 3$ dimensions. The AdS/CFT correspondence 'solves' these theories at large N. While rather little is understood about these theories at finite N ($N \neq 1$), it is known that they possess exactly flat directions along a Coulomb branch; a reflection of the ability of these N parallel branes to separate along transverse directions. This Coulomb branch is the metrically flat space $(R^8)^N/S_N$ or $(R^5)^N/S_N$, see [1]; the quotient by the symmetric group S_N reflects the identical nature of these branes.

In this note we study the radial quantization of (a sub class of) these Coulomb branch solutions; we pause to explain what this means. Quantum field theories on R^d are most often quantized by associating Hilbert Spaces with field configurations along constant time R^{d-1} slices. Under this procedure, the distinct M2 and M5 branes Coulomb branch solutions parameterize distinct superselection sectors. In this note we instead study the radial

JHEP12(2007)004

quantization of the worldvolume theories of M5 and M2 branes on R^6 and R^3 respectively. This procedure is equivalent to the quantization of these theories on $S^5 \times R$ or $S^2 \times R$, and is natural from several points of view. First, it introduces a mass gap into the system, regulating potentially severe infrared divergences. Relatedly (and more importantly for this note) it yields the dual to M theory on global, geodesically complete, $AdS_4 \times S^7$ and $AdS_7 \times S^4$ respectively.

Under radial quantization the world volume theories we study each have a unique vacuum. Distinct Coulomb branch configurations are normalizable, finite energy fluctuations about this vacuum.[1] Sub classes of these solutions are respectively $\frac{1}{8}$ and $\frac{1}{4}$ BPS. In this note we (radially) quantize these supersymmetric solutions and compute the partition functions Z_5^N and Z_2^N, over the resultant Hilbert Space. We conjecture[2] that Z_5^N and Z_2^N are the exact, finite N partition function over the 1/8 or 1/4 BPS Hilbert space of the theories we study.[3] Our results agree in particular with the spectrum of 'single trace' chiral primaries for the (0,2) theory of the M5 brane computed in [2].

We proceed to employ the AdS/CFT correspondence to gather evidence for our conjecture. In particular we demonstrate that

1. In the large N limit Z_5^N and Z_2^N reduce to the partition function over supersymmetric multi graviton configurations in $AdS_7 \times S^4$ and $AdS_4 \times S^7$ respectively.

2. The partition function obtained from the quantization of the full manifold of Mikhailov's supersymmetric giant gravitons ([4, 5] in $AdS_7 \times S^4$ and $AdS_4 \times S^7$ (along the lines of earlier studies [6, 7] agrees exactly with Z_5^N and Z_2^N. We also argue that the quantization multi dual giant gravitons may be regarded as bulk analogue of the quantization of the Coulomb Branch. A direct study of the quantization of dual giant gravitons (following previous studies [8, 3]) yields additional support for our formula for Z_5^N and Z_2^N.

The rest of this note is organized as follows. In section 2 we briefly review the supersymmetry algebra of the M5 and M2 brane world volume theories. We give a precise characterization of the supersymmetric states and the partition functions Z_5 and Z_2^N that we study in this note. We also compute the partition function over supersymmetric multi gravitons in $AdS_7 \times S^4$ and $AdS_4 \times S^7$. In section 3 we compute the partition function Z_5^N and Z_2^N by quantizing the relevant Coulomb branches. We argue that our procedure for obtaining Z_5^N and Z_2^N has a close bulk analogue in the quantization of N non interacting dual giant gravitons along the lines of [8, 3]. In section 4 we compute the partition function over appropriately supersymmetric giant gravitons in $AdS_7 \times S^4$ and $AdS_4 \times S^7$, using the methods of [6, 7]. Though this procedure is atleast superficially quite different from our quantization of the Coulomb branch of section 3, it gives exactly the same result. In section 5 we end with a brief discussion.

[1] They map to time dependent solutions of the relevant theories on $S^{d-1} \times R$.

[2] See [2] and [3] for related earlier work.

[3] Equivalently, we conjecture that the Coulomb Branch exhausts the set of appropriately supersymmetric 'classical' configurations in the full non abelian theory. In section 3 we demonstrate that the analogeous result is true for the low energy theory of D3-branes in IIB theory, i.e. $\mathcal{N} = 4$ Yang Mills theory.

JHEP12(2007)004

2. Supersymmetric states and the supergravity partition function

In this section we group theoretically characterize the supersymmetric states of interest to us in this note.[4] We also use the AdS/CFT correspondence to compute the partition function over these states in the large N limit.

2.1 M5 branes

The bosonic part of the supersymmetry algebra of the $(0,2)$ theory on the world volume of the M5 brane is $\mathrm{SO}(6,2) \times \mathrm{SO}(5)$. Supersymmetry generators (Qs) are simultaneously spinors of SO(5) and chiral spinors of $\mathrm{SO}(6) \in \mathrm{SO}(6,2)$. The Hermitian conjugates of these operators (Ss) are also SO(5) spinors and antichiral SO(6) spinors. Consider the set of four Qs with given a given set of SO(5) charges. For concretenes we choose these SO(5) charges to be $H_1 = H_2 = \frac{1}{2}$ where H_1 and H_2 are Cartan generators of SO(5) that generate rotations in orthogonal planes in an embedding R^5 and arbitrary SO(6) charges. These four supersymmetries, together with their Hermitian conjugates, generate the compact superalgebra SU(4/1).

We are interested in the SU(4/1) invariant states in the $(0,2)$ theory of N coincident M5-branes. There exists an equivalent characterization of these states; they are SO(6) singlets that simultaneously obey the BPS bound

$$E = 2H_1 + 2H_2. \tag{2.1}$$

In this note we will propose a formula for

$$Z_5^N = \mathrm{Tr}\, e^{-\mu_1 H_1 - \mu_2 H_2} \tag{2.2}$$

where the trace is taken over all SU(4/1) invariant states in the theory of N coincident M5 branes.

Let us immediately compute this partition function in the limit of infinite N. Recall that the Maldacena dual of the $(0,2)$ theory is M theory on $AdS_7 \times S^4$. In the infinite N limit all finite energy states are non interacting multi supergravitons. The spectrum of gravitons in $AdS_7 \times S^4$ was worked out and arranged into multiplets of the superconformal group in [10] and references therein (see also [11, 12]). All gravitons appear in short representations of the superconformal algebra. The primaries of these representations are SU(4) singlets, tranform in the n^{th} symmetric traceless representation of SO(5), and have energy equal to $2n$. It turns out that the only SU(4/1) invariant states in these representatations are those primary states with charges that obey the equation $H_1 + H_2 = n$. There exists one such state for every partitioning of n into a sum of two non negative integers. Summing over all primaries (and hence over all n) we conclude that the set of supergravitons of interest to us are labeled by two non negative integers (n_1, n_2) that cannot simultaneously be zero; the corresponding states have charges $(H_1, H_2) = (n_1, n_2)$. The large N fixed chemical potential limit of the SU(4/1) invariant subsector of the $(0,2)$

[4] See for instance [9] for information about these superalgebras and their unitary representations.

theory is simply the Fock space over these supersymmetric graviton states. The partition function over this Fock space is given by

$$Z_5^\infty = \mathrm{Tr}_{\mathcal{H}_\infty}\, e^{-\mu_1 H_1 - \mu_2 H_2} = \prod_{n_1,n_2=0}^{\infty} \frac{1}{1 - e^{-n_1\mu_1 - n_2\mu_2}} \tag{2.3}$$

where the case $n_1 = n_2 = 0$ is excluded from the product.

2.2 M2 Branes

The bosonic part of the supersymmetry algebra of the world volume of the M2 brane is $\mathrm{SO}(3,2) \times \mathrm{SO}(8)$. Supersymmetry generators (Qs) and their complex conjugates Ss are both simultaneously spinors of $\mathrm{SO}(3) \in \mathrm{SO}(3,2)$ and chiral spinors of SO(8). Consider the set of 2 Qs with arbitrary SO(3) charges and a given set of SO(8) charges. For concreteness we choose these to be $H_1 = H_2 = H_3 = H_4 = \frac{1}{2}$ where H_i, $i = 1 \ldots 4$ are Cartan generators of SO(8) that generate rotations in mutually orthogonal planes in R^8. These two supersymmetries, together with their Hermitian conjugates, generate the compact superalgebra SU(2/1). SU(2/1) invariant states admit an alternate characterization; they are SO(3) singlets that simultaneously obey the BPS bound

$$E = \frac{1}{2}\left(H_1 + H_2 + H_3 + H_4\right). \tag{2.4}$$

In this note we propose a formula for

$$Z_2^N = \mathrm{Tr}\, e^{-\sum_{i=1}^4 \mu_i H_i} \tag{2.5}$$

over all such states.

The Maldacena dual of the M2-brane theory is M theory on $AdS_4 \times S^7$. The spectrum of gravitons on $AdS_4 \times S^7$ was worked out in [13]. This spectrum was arranged in representations of the superconformal algebra in [14] (see also [11, 12]). All gravitons lie in supershort representations of the superconformal algebra. The primaries of these representations are SO(3) scalars, transform in traceless symmetric tensors (of arbitrary rank n) of SO(8) and have energy given by $\frac{n}{2}$. It turns out that the only SU(2/1) invariant states in these multiplets are those primary states whose charges obey $\sum_{i=1}^4 H_i = n$. Suming over all primaries (and so all n) we conclude that the set of SU(2/1) invariant gravitons on this space is labeled by four non negative integers that cannot all be zero. In terms of these integers $(H_1, H_2, H_3, H_4) = (n_1, n_2, n_3, n_4)$ where n_i ($i = 1 \ldots 4$). It follows that the partition function over multi super gravitons is given by

$$Z_2^\infty = \mathrm{Tr}_{\mathcal{H}_\infty}\, e^{-\mu_1 H_1 - \mu_2 H_2 - \mu_3 H_3 - \mu_4 H_4} = \prod_{n_1,n_2,n_3,n_4=0}^{\infty} \frac{1}{1 - e^{-n_1\mu_1 - n_2\mu_2 - n_3\mu_3 - n_4\mu_4}} \tag{2.6}$$

where the term with all n_i zero is excluded from the product.

3. Quantization of the Coulomb branch

In this section we determine the partition function Z_5^N and Z_2^N by radially quantizing Coulomb branch configurations of the world volume theories of $M5$ and $M2$ branes. We begin this section with a brief discussion of the radial quantization of conformally coupled scalar field theories in arbitrary dimension.

3.1 Radial quantization of a free scalar fields in aribtrary dimension

Consider a free, complex, conformally coupled d dimensional scalar field on a unit S^{d-1}

$$S = \int dt d^{d-1}x \left(\partial\phi_S . \partial\phi_S^* - \frac{d-2}{2}\, \phi_S \phi_S^* \right). \tag{3.1}$$

where ϕ_S^* is the complex conjugate of ϕ_S. Analytically continuing to Euclidean time and conformally mapping to the R^d we find the Euclidean action

$$S = \int \partial\bar{\phi} . \partial\phi \tag{3.2}$$

where

$$\phi(x) = \frac{\phi_S(x)}{|x|^{\frac{d-2}{2}}}, \qquad \bar{\phi}(x^\mu) = \frac{\phi_S^*(\frac{x^\mu}{x^2})}{|x|^{\frac{d-2}{2}}}. \tag{3.3}$$

Notice that

$$\bar{\phi}(x) = |x|^{-(d-2)} \phi^*(x^\mu/|x|^2); \tag{3.4}$$

in particular $\bar{\phi}$ is not simply the complex conjugate of ϕ. It is possible to check that the equations of motion

$$\partial^2\phi = \partial^2\bar{\phi} = 0. \tag{3.5}$$

and (3.4) are mutually consistent.[5]

Regular solutions of (3.1) map to field configurations ϕ and $\bar{\phi}$ whose singularities are localized at $x = 0$ and $x = \infty$.[6] The general solution of (3.5) and (3.4) is given by

$$\begin{aligned} \phi &= \sum c^*_{\mu_1 \ldots \mu_m} x^{\mu_1} \ldots x^{\mu_m} + y^{d-2} \sum d_{\nu_1 \ldots \nu_n} y^{\nu_1} \ldots y^{\nu_n} \\ \phi &= \sum d^*_{\nu_1 \ldots \nu_m} x^{\nu_1} \ldots x^{\nu_n} + y^{d-2} \sum c_{\mu_1 \ldots \mu_m} y^{\mu_1} \ldots y^{\mu_n} \\ \text{where} \quad y^\mu &= \frac{x^\mu}{x^2} \end{aligned} \tag{3.6}$$

and $c_{\mu_1 \ldots \mu_m}$ and $d_{\mu_1 \ldots \mu_m}$ are arbitrary complex traceless symmetric tensors.

It follows from (3.4) (and we note for future reference) that

$$\begin{aligned} \partial_\mu \bar{\phi} &= \left(\frac{1}{x^{d-2}}\right) \left[-(d-2)\frac{x_\mu}{x^2}\phi^* - 2\frac{x_\mu x^\alpha \partial_\alpha \phi^*}{x^4} + \frac{\partial_\mu \phi^*}{x^2} \right] \\ x.\partial\bar{\phi} &= -\left(\frac{1}{x^{d-2}}\right) \left[(d-2)\phi^* + \frac{x.\partial\phi^*}{x^2} \right] \\ \partial\phi.\partial\bar{\phi} &= \left(\frac{1}{x^{d-2}}\right) \left[-(d-2)\frac{x.\partial\phi\phi^*}{x^2} - 2\frac{(x.\partial\phi)(x.\partial\phi^*)}{x^4} + \frac{\partial\phi.\partial\phi^*}{x^2} \right]. \end{aligned} \tag{3.7}$$

[5] This follows from the observation that $\partial_x^2(y^{d-2}\chi) = y^{d+2}\partial_y^2\chi$ where ∂_x^2 and ∂_y^2 are the Laplacian with respect to the x and $y^\mu = x^\mu/|x|^2$ respectively.

[6] The coefficients of solutions regular at zero turn into creation operators , while the coefficients of solutions regular at infinity turn into destruction operators upon quantization.

The symplectic form from (3.1)

$$\omega = \int_{S^{(d-1)}} \left[d\dot{\phi_S} \wedge d\phi_S^* + d\dot{\phi_S^*} \wedge d\phi_s \right] \tag{3.8}$$

translates into

$$\begin{aligned} -i\omega = & \int_{S^{d-1}} d\left[(x.\partial) \left(x^{\frac{(d-2)}{2}} \phi(x^\mu) \right) \right] \ \wedge \ d\left[x^{\frac{d-2}{2}} \phi^*(\frac{x^\mu}{x^2}) \right] \\ & + d\left[(x.\partial) \left(x^{\frac{-(d-2)}{2}} \phi^*(\frac{x^\mu}{x^2}) \right) \right] \ \wedge \ d\left[x^{\frac{(d-2)}{2}} \phi(x^\mu) \right]. \end{aligned} \tag{3.9}$$

The generator of time translations in (3.1) maps to the generator of scale transformations in (3.2). The corresponding conserved current is given by

$$D_\mu = x_\mu \left(\partial\bar{\phi}.\partial\phi \right) - \frac{(d-2)}{2} [\partial_\mu \phi \bar{\phi} + \partial_\mu \bar{\phi} \phi] - \partial_\mu \phi (x.\partial\bar{\phi}) - \partial_\mu \phi (x.\partial\bar{\phi}), \tag{3.10}$$

and its conserved charge E (the energy) is given by

$$\begin{aligned} E = & \int |x|^{(d-2)} x.D d\Omega_{(d-1)} \\ = & \int \partial\phi.\partial\bar{\phi} - 2(x.\partial\phi)(x.\partial\bar{\phi}) - \frac{d-2}{2} [x.\partial\phi\bar{\phi} + x.\partial\bar{\phi}\phi] \\ = & \int \partial\phi.\partial\phi^* + \left(\frac{d-2}{2} \right) [(x.\partial\phi)\phi^* + (x.\partial\phi^*)\phi] + \frac{(d-2)^2}{2} \phi\phi^* \end{aligned} \tag{3.11}$$

where the integrals in the last two lines are evaluated on the unit S^{d-1} $|x| = 1$.

The action (3.2) is invariant under a $\mathrm{U}(1)$ scaling of the field ϕ. The corresponding Noether current, H_μ is given by

$$H_\mu = \bar{\phi} \partial_\mu \phi - \phi \partial_\mu \bar{\phi}. \tag{3.12}$$

The associated conserved charge H is given by

$$\begin{aligned} H = & \int d\Omega_{(d-1)} x^{(d-2)} x.H \\ = & \int \phi^* x.\partial\phi + \phi x.\partial\phi^* + (d-2)\phi\phi^* \end{aligned} \tag{3.13}$$

where the integral in the last line is evaluated on the on the unit sphere $|x| = 1$.

Note that

$$\begin{aligned} E - \frac{(d-2)}{2} H = & \int \partial\phi.\partial\phi^* \\ E + \frac{d-2}{2} H = & \int \partial\phi.\partial\phi^* + (d-2)[(x.\partial\phi)\phi^* + (x.\partial\phi^*)\phi] + (d-2)^2 \phi\phi^* \\ = & \int \partial\bar{\phi}.\partial\bar{\phi}^* \end{aligned} \tag{3.14}$$

It follows that $(E \pm \frac{(d-2)}{2}H)$ is zero if and only if respectively ϕ or $\bar{\phi}$ are constant. Further

$$E = \frac{1}{2}\int \left(\partial\phi.\partial\phi^* + \partial\bar{\phi}.\partial\bar{\phi}^*\right). \tag{3.15}$$

Consequently, in order for a configuration to have zero energy, ϕ and $\bar{\phi}$ must each be constant. However it follows from (3.4) that ϕ and $\bar{\phi}$ can both be constant only if they are both zero; thus $\phi = \bar{\phi} = 0$ is the only zero energy configuration.

Using (3.8), (3.10) and (3.12) it is a simple matter to quantize the general solution (3.6) to obtain the spectrum of energies and charges. We illustrate this on a subclass of solutions. Let z represent any complex direction in R^d and consider the subclass of solutions

$$\phi = \sum_n \frac{c_n^*}{\sqrt{K_n}} z^n. \tag{3.16}$$

where the constant K_n is chosen (for convenience) to be

$$\begin{aligned} K_n &= 4\pi n \Omega_{d-3} \int_0^1 r^{2n-1}(1-r^2)^{\frac{d-4}{2}} dr \\ &= 2\pi(d-2+2n)\Omega_{d-3} \int_0^1 r^{2n+1}(1-r^2)^{\frac{d-4}{2}} dr. \end{aligned} \tag{3.17}$$

We find

$$\begin{aligned} H &= \sum_n |c_n|^2 \\ E - \frac{d-2}{2}H &= \sum_n n|c_n|^2 \\ \omega &= i\sum_n dc_n^* \wedge dc_n \end{aligned} \tag{3.18}$$

The quantization of (3.18) turns c_n^* and c_n respectively into creation and annihilation operators of unit charge and energy $n + \frac{d-2}{2}$ respectively.

The quantization of the general solution (3.6) proceeds along similar lines. $c_{\mu_1\ldots\mu_n}$ and $c^*_{\mu_1\ldots\mu_n}$ turn into operators that are proportional to unit normalized creation and annihilation operators of energy $n+\frac{d-2}{2}$ and unit $\mathrm{U}(1)$ charge under quantization. Similarly $d_{\mu_1\ldots\mu_m}$ and $d^*_{\mu_1\ldots\mu_m}$ become creation and annihilation operators of energy $m + \frac{d-2}{2}$ and negative unit $\mathrm{U}(1)$ charge respectively.

Real scalar fields obey the constraint

$$\psi = \bar{\psi}. \tag{3.19}$$

The quantization of these theories proceeds along similar lines. The equation (3.15) also applies to real scalar fields with an extra factor of $\frac{1}{2}$ on the r.h.s. in standard real normalization (note also that the two terms on the r.h.s. of this equation are equal in this special case). As it is impossible for a nonzero constant value of ψ to obey (3.19) (see (3.4)) it follows that $\psi = 0$ is the only field configuration with vanishing energy.

3.2 Quantization of the BPS sector of the U(1) theory

A single M2 brane has four free complex scalar fields, ϕ_i $(i = 1 \dots 4)$ on its world volume. The field ϕ_i has charge δ_i^j under the Cartan U(1) rotations H_j of section 2. The BPS combinaton (3.14) evaluates to

$$E - \frac{1}{2}\sum_{i=1}^{4} H_i = \sum_{i=1}^{4} \int \partial\phi_i . \partial\phi_i^* \tag{3.20}$$

It follows that the constant field configurations $\phi_i(x) = \frac{\phi_i}{\sqrt{2\pi}}$ are the only classical BPS configurations; the corresponding symplectic form, energy and charge formulae are

$$\begin{aligned} \omega &= i \sum_{i=1}^{4} d\phi_i \wedge d\phi_i^* \\ E &= \frac{1}{2} \sum_{i=1}^{4} \phi_i \phi_i^* \\ H_i &= \phi_i \phi_i^* . \end{aligned} \tag{3.21}$$

The quantization of (3.21) yields the Hilbert space of a four dimensional harmonic oscillator; charge operator H_i turns into the number operator of the i^{th} oscillator on this space.

A single M5 brane has two complex free scalar fields, ϕ_i $(i = 1 \dots 2)$ and one real scalar field ψ on its world volume. The complex fields ϕ_i have charges δ_i^j under the charge H_j; ψ is uncharged. We have

$$E - \frac{1}{2}\sum_{i=1}^{2} H_i = \frac{1}{2} \partial\psi . \partial\psi^* + \sum_{i=1}^{2} \int \partial\phi_i . \partial\phi_i^* . \tag{3.22}$$

The set of classically BPS configurations is given by constant $\phi^i(x) = \frac{\phi^i}{\sqrt{(2\pi)^3}}$ and $\psi = 0$ (recall that it is impossible for a nonzero constant ψ to obey (3.19)). On this class of solutions

$$\begin{aligned} \omega &= i \sum_{i=1}^{2} d\phi_i \wedge d\phi_i * \\ E &= 2 \sum_{i=1}^{2} \phi_i \phi_i * \\ H_i &= \phi_i \phi_i^* \end{aligned} \tag{3.23}$$

The quantization of (3.23) yields the Hilbert space of a two dimensional harmonic oscillator; the charge operator H_i is the number operator of the i^{th} oscillator on this space.

In summary, the symplectic manifold of classically SU(4/1) and SU(2/1) invariant configurations on the theory of a single M5 or M2 brane is C^2 and C^4 respectively. The quantization of these manifolds yields the the Hilbert space of two and four dimensional harmonic oscillators, respectively.

3.3 Quantization of the Coulomb branch

As we have remarked in the introduction, the world volume theory of N M5 or M2 branes possesses Coulomb branches. Away from their singularities, these Coulomb branches are metrically flat; the spaces in question are $(R^8)^N/S_N$ and $(R^5)^N/S_N$ respectively. Using the results of the previous section, it follows that classically SU(4/1) or SU(2/1) invariant Coulomb branch solutions constitute the symplectic manifolds $(C^4)^N/S_N$ and $(C^2)^N/S_N$ respectively. The quantization of these manifolds yields fock space of N identical, non interacting bosons in a 4 or 2 dimensional harmonic oscillator potential. The partition function for this system follows immediately from the usual formulas of Bose Statistics;

$$\sum_{m=1}^{\infty} p^N Z_5^N = \prod_{n_1,n_2=0}^{\infty} \frac{1}{1-pe^{-n_1\mu_1-n_2\mu_2}} \tag{3.24}$$

and

$$\sum_{m=1}^{\infty} p^N Z_5^N = \prod_{n_1,n_2,n_3,n_4=0}^{\infty} \frac{1}{1-pe^{-n_1\mu_1-n_2\mu_2-n_3\mu_3-n_4\mu_4}} \tag{3.25}$$

3.4 Absence of additional supersymmetric configurations in $\mathcal{N}=4$ Yang Mills

As we have explained in the previous subsection, the world volume theory of M5 and M2 branes has a class of, respectively, SU(4/1) and SU(2/1) invariant configurations on their Coulomb branch. We conjecture that these configurations are exhaustive; that the full non ablelian world volume theories have possess no further SU(4/1) or SU(2/1) invariant configurations, so that (3.25) and (3.24) represent the exact partition functions over SU(4/1) and SU(2/1) invariant states in these theories.

Our poor understanding of the structure of the non abelian theory on M2 and M5 branes makes it difficult to directly verify this conjecture. In this sub section we will, however, demonstrate that the analogous claim is indeed true of much better understood world volume theory of N D3 branes - U(N) $\mathcal{N}=4$ Yang Mills theory.

The $1/8^{\text{th}}$ BPS sector of $\mathcal{N}=4$ Yang Mills theory has recently been studied in some detail in [15, 7, 8]; we will not pause here to characterize this subsector group theoretically, but instead refer to reader to [15] for such details. This worldvolume theory possesses 3 complex adjoint valued scalar fields $\phi_i = \psi_{2i-1} + i\psi_i$ (where $\psi_i, \quad i = 1 \ldots 6$ are Hermitian $N \times N$ matrix valued scalar fields). The restriction of this sector to states (or operators) made entirely out of scalars[7] analogous, in many ways, to the supersymmetric states we have been studying on M5 and M2 brane world volumes; this subclass of $1/8^{\text{th}}$ states consist of Lorentz scalars that obey the BPS bound $E = \frac{4-2}{2}(H_1+H_2+H_3) = H_1+H_2+H_3$ where H_i refer to the three Cartan generators of the SO(6) R symmetry of this theory. The

[7]It should be possible to construct the full $\frac{1}{8}^{\text{th}}$ BPS cohomology of Yang Mills theory (studied in [15]) from the radial quantization of supersymmetric fermionic field configurations, together with the scalars studied in this paper. We expect the relevant fermionic configurations to be constant diagonal gaugino fields. As these configurations have no analogues in the theory of M5 and M2 branes, we do not perform a detailed study of these configurations in this paper.

JHEP12(2007)004

Euclidean Lagrangian of this theory - restricted to configurations over which only scalar fields are nonzero - is given by

$$\frac{1}{g_{\rm YM}^2}\int {\rm Tr}\left(\sum_i \partial\phi_i\partial\bar{\phi}_i + \frac{1}{2}\sum_{i\neq j}[\psi_i,\psi_j][\psi_j,\psi_i]\right) \tag{3.26}$$

(recall ψ_i are the real components of the complex scalars ϕ_i). Performing the radial quantization of this Lagrangian, imitating the work out of subsection 3.1, we find

$$E - \sum_{i=1}^{3} H_i = \frac{1}{g_{\rm YM}^2}\,{\rm Tr}\int\left(\sum_{i=1}^{3}\partial\phi_i.\partial\phi_i^\dagger + \sum_{m,n=1}^{6}[\psi_m,\psi_n][\psi_n,\psi_m]\right). \tag{3.27}$$

It follows from (3.27) that the set of scalar BPS configurations of this theory consist of constant diagonal matrices ϕ_i, further gauge invariance requires us to identify matrices with permuted eigenvalues[8] (see [17] for closely related remarks). This set of matrices parameterizes the Coulomb branch studied above. We conclude that the set of 1/8 BPS configurations in $\mathcal{N}=4$ Yang Mills lies entirely within the Coulomb branch;[9] in other words the diagonal supersymmetric configurations that generalize those of [16] constitutes the full set of scalar $1/8^{\rm th}$ BPS configurations of Yang Mills theory.

3.5 Dual giants: bulk duals of the Coulomb branch

The bulk AdS dual of an M5 or M2 brane on its Coulomb Branch is an M5 or M2 brane that is puffed out in AdS_7 or AdS_4 in an SO(6) or SO(3) invariant manner. The SO(6) or SO(3) invariant configurations in AdS_7 or AdS_4 are a one parameter set of 5 and 2 dimensional spheres that foliate spatial sections of AdS.[10] Branes wrapping such spheres, and otherwise moving on S^4 or S^7 in a supersymmetric fashion, have been identified [16, 18] and studied in detail; they are called dual giant gravitons. In this subsection we review the construction of these configurations [18, 16] and their quantization (partially performed in [3]; see also [19, 20, 8] for related work). We find that the result of the quantization of a single supersymmetric dual giant graviton is identical to the quantization of a single brane on the Coulomb Branch discussed in the previous section. Our discussion of the quantization of dual giant gravitons overlaps with the in the case of the M2 brane..

[8] Mapping to S^3 we recover a slight generalization of the diagonal solutions considered in [16].

[9] The quantization of these configurations may be carried out; the relevant formulae are

$$\begin{aligned}\omega &= i\sum_{i=1}^{3}{\rm Tr}\, d\phi_i\wedge d\phi_i^\dagger\\ E &= \sum_{i=1}^{3}{\rm Tr}\,\phi_i\phi_i^\dagger\\ H_i &= {\rm Tr}\,\phi_i\phi_i^\dagger\end{aligned} \tag{3.28}$$

[10] Exceptional orbits consist of points (fixed points under the rotational action). These orbits may be regarded as the zero radius limit of our spheres, and are automatically taken into account below.

Let us first describe dual giant gravitons in $AdS_{m+2} \times S^{n+2}$ [18, 16]. Although backgrounds of this form are known to appear in string theory only for specific values of m and n, in this section and the next we will leave m and n arbitrary. The formulae we present below apply to M5, M2, and D3 brane dual giant gravitons upon setting (m, n) to $(5, 2)$, $(2, 5)$ and $(3, 3)$ respectively.

Let the metric on $AdS_{m+2} \times S^{n+2}$ be given by

$$d\tilde{s}^2 = R_1^2 \left(-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega_m^2 \right) + R_2^2 d\Omega_{n+2}^2 \tag{3.29}$$

where Ω_k is the metric on the unit k sphere. Below we will find it convenient to work with the scaled metric

$$ds^2 = \frac{d\tilde{s}^2}{R_1^2} = G_{\mu\nu} dx^\mu dx^\nu. \tag{3.30}$$

$R = \frac{R_2}{R_1}$ is the radius of the sphere in the new rescaled metrics (3.30); in the solutions that appear in string theory $R = \frac{2}{m-1} = \frac{2}{d-2}$ where d is the spacetime dimension of the dual world volume theory. In particular $R = \frac{1}{2}$ in $d = 6$ and $R = 2$ in $d = 3$. The space (3.29) has a nonzero $n+1$ form gauge potential A_{n+1} turned on such that the corresponding field strenth preserves the symmetry of S^{n+2} and $\int_{S^{n+2}} dA_{n+1} = 2\pi N$. We denote the $m+1$ form dual to A_{n+1} by A_{m+1}.

The action for an m brane propagating in this background is given by

$$S = -\frac{N^{\frac{m-1}{2}} 2^{\frac{m-3}{2}}}{\Omega_m} \int \sqrt{-g} + \int A_{m+1}. \tag{3.31}$$

For furture use we note that the action for an n brane propagating in the same background is given by

$$S = -\frac{N}{R^{n+1} \Omega_n} \int \sqrt{-g} + \int A_{n+1}. \tag{3.32}$$

$g_{\alpha\beta}$ is the pullback of the metric $G_{\mu\nu}$ (see (3.30)) on the world volume of the brane in question.

As we have explained above, in this subsection we are interested in an m-brane that completely wraps the m dimensional sphere in AdS_{m+2} and so is effectively a particle in the remaining $n+4$ spacetime dimensions. The degrees of freedom of this particle are the $n+2$ position coordinates in S^{n+2} and its location in the radial ρ direction in AdS_{m+2}.

We will find it convenient to make a particular choice of coordinates on S^{n+2}. Regarding S^{n+2} as the unit sphere in R^{n+3} we set

$$x_1 + i x_2 = r_1 e^{i\theta_1}, \quad \ldots, \; x_{n+2} + i x_{n+3} = r_{\frac{n+3}{2}} e^{i\theta_{\frac{n+3}{2}}} \tag{3.33}$$

when n is odd and

$$x_1 + i x_2 = r_1 e^{i\theta_1}, \quad \ldots x_{n+1} + i x_{n+2} = r_{\frac{n+2}{2}} e^{i\theta_{\frac{n+2}{2}}}, \quad x_{n+3} = h \tag{3.34}$$

when n is even. The metric on S^{n+2} is

$$d\Omega_{n+2}^2 = dh^2 + \sum_{i=1}^{k} {dr_i}^2 + \left(d\sqrt{1 - h^2 - \sum_{i=1}^{k} r_i^2} \right)^2 + \left(1 - \sum_{i=1}^{k} {r_i}^2 - h^2 \right) {d\theta_{k+1}}^2 + \sum_{i=1}^{k} {r_i}^2 {d\theta_i}^2 \tag{3.35}$$

where $k = \frac{n+1}{2}$ when n is odd (in which case we simply set h to zero in (3.35)) and $k = \frac{n}{2}$ when n is even.

The action for the dual giant graviton is given by

$$S = 2^{\frac{m-3}{2}} N^{\frac{m-1}{2}} \int d\tau \; \mathcal{L} \tag{3.36}$$

where

$$\mathcal{L} = +\dot{t} \sinh^{m+1} \rho - \sinh^m \rho \sqrt{\dot{t}^2 \cosh^2 \rho - \dot{\rho}^2 - R^2 \left(\frac{d\Omega_{n+2}}{d\tau} \right)^2} \tag{3.37}$$

where $(\frac{d\Omega_{n+2}}{d\tau})^2$ is the sigma model kinetic term on a unit S^{n+2} and, once again, h is simply set to zero when n is odd.

In order to proceed we choose $\tau = t$. As the variables θ_i do not appear in the lagrangian, $\dot{\theta}_i$ are constants on all solutions to the equations of motion. Let us study the ansatz h, r_i, $\dot{\theta}_i$ and $\rho =$ all constant. This ansatz yields solutions when $h = 0$ (forced by the h equation of motion), $\dot{\theta}_i$ are all equal (forced by the r_i equation of motion) and $\dot{\theta}_i = \frac{1}{R}$ (forced by the ρ equation of motion). Note that r_i, ρ and the intial values of θ_i are unconstrained, and parameterize distinct solutions. On solutions H_i is the momentum conjugate to θ_i while the energy is the negative of the momentum conjugate to t; we find

$$\sum_{i=1}^{k+1} H_i = R(\sinh^{m-1} \rho) \sum_{i=1}^{k+1} {r_i}^2 = R \sinh^{m-1} \rho$$
$$E = \sinh^{m-1} \rho = \frac{1}{R} \sum_{i=1}^{k+1} H_i. \tag{3.38}$$

As a consequence of the last equation in (3.38) the solutions described in the previous paragraph are all 'supersymmetric'.

Notice that h (which exist only for even n) is always zero on solutions to the equations of motion. This is a dual expression of fact that the real field ψ of section 3.2 is zero on 'supersymmetric' Coulomb branch configurations.

On these solutions the momentum corresponding to ρ ,r_i and h all vanish. The momentum conjugate to θ_i is given by

$$P_{\theta_i} = (\sinh^{m-1} \rho) R^2 {r_i}^2 \dot{\theta}_i = R {r_i}^2 \sinh^{m-1} \rho. \tag{3.39}$$

It follows that

$$\omega = \sum_i dP_{\theta_i} \wedge d\theta_i = \sum_{i=1}^{k+1} 2R_i dR_i \wedge d\theta_i = i \sum_{i=1}^{k+1} d\phi_i \wedge d\phi_i^* \tag{3.40}$$

where

$$R_i = \sqrt{R}(\sinh^{\frac{m-1}{2}} \rho) r_i \tag{3.41}$$

and

$$\phi_i = R_i e^{-\theta_i}. \tag{3.42}$$

The expressions for charges are[11]

$$\begin{aligned} E &= \frac{d-2}{2} \sum_i |\phi_i|^2 \\ H_i &= |\phi_i|^2. \end{aligned} \tag{3.43}$$

Notice that (3.40) and (3.43) are identical to (3.23). It follows that the quantization of a single dual giant graviton is identical to the quatization of a single brane on the Coulomb branch.[12] The interesting translation formulas (3.41) and (3.42) provide a dictionary to convert between spacetime coordinates and Yang Mills field expectation values.

4. Supersymmemtric states from giant gravitons

In the previous section we have determined the partition function over supersymmetric states on the M5 brane or M2-brane world volume by quantizing supersymmetric Coulomb branch configurations. In this section we will demonstrate that there exists a bulk quantization - superficially unrelated to the quantization of the Coulomb branch - that permits relatively simple finite N computation the spectrum of supersymmetric states in the world volume theories of M5 and M2 branes. The procedure we refer to is the quantization of giant gravitons.

Over 5 years ago Mikhailov constructed all SU(4/1) invariant giant graviton configurations in $AdS_7 \times S^4$ and $AdS_4 \times S^7$. Mikhailov's giant gravitons all sit at the point $\rho = 0$ in AdS_7 or AdS_4 (i.e. at the fixed point of the SO(6) or SO(3) killing symmetry in this space). The shape of these probes with the S^4 or S^7 is described by an indirect construction. Let us first consider the case of S^4. Let S^4 be embedded as the unit sphere in R^5. The rotations H_1 and H_2 are represented by killing vector fields corresponding to rotations in orthogonal two planes in R^5. A rotation by angle $\frac{\pi}{2}$ simultaneously in each of the H_1 and H_2 directions yields a complex structure in $C^2 \in R^5$. Let z_1 and z_2 be complex coordinates in this C^2. Mikhailov has demonstrated that M2-branes that wrap the intersection of the 'holomorphic' surface $F(e^{-it}z_1, e^{-it}z_2) = 0$ with the unit 5 sphere are all SU(4/1) invariant. Similarly, supersymmetric giant gravitons in $AdS_4 \times S^7$ are M5 branes that sit at the centre of AdS_4. Let S^7 be regarded as the unit sphere in R^8. H_i $(i = 1 \ldots 4)$ are rotations in mutually orthogonal planes in R^8. A simultaneous rotation by $\frac{\pi}{2}$ in each of these directions is a complex structure on this space. Mikhailov has demonstrated that M2-branes that wrap the intersection of the 'holomorphic' surface $F(e^{-it}z_1, e^{-it}z_2, e^{-it}z_3, e^{-it}z_4) = 0$ with the unit 5 sphere are all SU(4/1) invariant.

[11] Similar results are reported in [3].

[12] See [8] for arguments that suggest this identity should persist for collections of multi giant gravitons.

We now turn to the quantization of giant gravitons. Following [7] we first regulate the space of holomorphic functions on C^2 or C^4. Let P_d^2 or P_d^4 refer to the set of holomorphic polynomials in C^2 or C^4 whose degree less than or equal to d. P_d is a linear vector space of dimensionality $n_d = \binom{d+2}{2}$ or $\binom{d+4}{4}$. In this section we demonstrate that as a symplectic manifold, P_1^2 and P_1^4 are respectively the spaces CP^2 or CP^4 with symplectic form (derived by restricting the dynamical symplectic form from the Born Infeld and Wess Zumino action to our subspace of solutions) $N\omega_{\rm FS}$ where $\omega_{\rm FS}$ is the Fubini Study form on CP^2 or CP^4. It follows from this result, together with the formal arguments of sections 2-4 in [7], (all of which apply to the context of this note) that P_d is the symplectic manifold CP^{n_d-1} with a current (distributional) symplectic form in the cohomology class of $N\omega_{\rm FS}$ (see [7]). In particular the partition function over the Hilbert space obtained from the quantization of this symplectic manifold is (see section 4 of [7]) the power of p^N in

$$\prod_{n_1,n_2=0}^{n_1+n_2\leq d} \frac{1}{1-pe^{-n_1\mu_1-n_2\mu_2}} \tag{4.1}$$

where

$$Z_5^N = \mathrm{Tr}\, e^{-\mu_1 H_1-\mu_2 H_2} \tag{4.2}$$

for the case of the M5 brane, and in

$$\prod_{n_1,n^2,n_3,n_4=0}^{n_1+n_2+n_3+n_4\leq d} \frac{1}{1-pe^{-n_1\mu_1-n_2\mu_2-n_3\mu_3-n_4\mu_4}} \tag{4.3}$$

where

$$Z_2^N = \mathrm{Tr}\, e^{-\mu_1 H_1-\mu_2 H_2-\mu_3 H_3-\mu_4 H_4}. \tag{4.4}$$

for the case of the M2 brane. (4.1) and (3.25) are the partition functions obtained from the quantization of regulated classes of Mikhailov's solutions. In order to remove the regulator we simply take the limit $d \to \infty$; under this limit (4.1) and (4.3) reduce to our proposals (3.24) and (3.25) for the partition functions over appropriate states in the M5 and M2 brane theories respectively.

In the rest of this section we will construct the manifold P_1^2 and P_1^4 of 'linear' Mikhailov solutions.

4.1 Quantization of linear polynomials

In this section we construct and quantize the intersections of

$$\sum_{i=1}^{k+1} e^{-i\frac{t}{R}} c_i \frac{y_i}{R} - 1 = 0 \tag{4.5}$$

in C^k (here y_is are the coordinates on C^k) with

(a) with the sphere of radius R in C^{k+1} and

(b) with the sphere of radius R in R^{2k+3}.

We follow the procedure of [7]; see also [19, 20] for related work.

We will quantize our manifold of solutions with respect to the canonical symplectic form obtained from the action (3.32) (recall that (3.32) describes the motions of probe M5 branes in $AdS_4 \times S^7$, probe M2 branes in $AdS_7 \times S^4$ and probe D3 branes in $AdS_5 \times S^5$ for appropriate values of the parameters m and n defined before (3.32); the sphere referred to above may be identified with the sphere in (3.30)).

The action (3.32) may be written out in more detail as

$$\mathcal{S} = \frac{N}{R^{n+1}\Omega_n} \int \sqrt{-g}\; d^n\sigma\; dt + \int_{S^{n+2}} d^n\sigma A_{\mu_0\mu_1\cdots\mu_n} \dot{x^{\mu_0}} \frac{\partial x^{\mu_1}}{\partial \sigma_1} \cdots \frac{\partial x^{\mu_n}}{\partial \sigma_n} \tag{4.6}$$

Here x^μs are coordinates on S^{n+2} and σs are coordinates on the world volume of the brane Σ^n. g_{ij} is the induced metric on Σ^n. A is the $(n+1)$ form such that $dA = F = \frac{2\pi N}{R^{n+2}\Omega_{n+2}}\epsilon$ where ϵ is the volume form on the S^{n+2}. We will find it convenient to work with the rescaled complex variables $z_i = \frac{y_i}{R}$. By a $\mathrm{U}(k+1)$ rotation any linear polynomial of the form (4.5) can be expressed as

$$c_0 e^{-i\frac{t}{R}} z_{k+1} - 1 = 0 \tag{4.7}$$

where $|c_0|^2 = \sum_i |c_i|^2$. The intersection of (4.7) with an S^{n+2} of radius R, centered about the origin, is an S^n of radius $R\sqrt{1 - \frac{1}{|c_0|^2}}$. It is not difficult to check that (4.7) satisfies the equations of motion that follow from (4.6). Further, the energy (momentum conjugate to $-t$) and the angular momenta on this solution are (see appendix A)

$$\begin{aligned} E &= \frac{N}{R}\left(1 - \frac{1}{|c_0|^2}\right)^{\frac{n-1}{2}} \\ P_\theta &= N\left(1 - \frac{1}{|c_0|^2}\right)^{\frac{n-1}{2}} \\ E &= \frac{1}{R} P_\theta \end{aligned} \tag{4.8}$$

Let us now return to the general linear solution (the solution parameterized by arbitrary complex coeficeints c_i). The symplectic form is a closed two-form on the parameter space formed by the coefficients (c_i) of the solution and the symplectic form must respect the $\mathrm{U}(k+1)$ invariance of the action. These conditions constrain the symplectic form to be of the form

$$\omega = f(|c|^2)\; \left(\frac{d\bar{c^i} \wedge dc_i}{2i}\right) + f'(|c|^2)\; \bar{c^i} c_j\; \left(\frac{d\bar{c^j} \wedge dc_i}{2i}\right) \tag{4.9}$$

This function f can be expressed as a sum of two terms one coming from the Born-Infeld part of the action (denoted by f_{BI}) and one coming from the Wess-Zumino term (denoted by f_{WZ}). Below (in most of the rest of this section) we compute f_{BI} and f_{WZ} following [7].

We find

$$\begin{aligned} f_{\rm BI}(x) &= -2N \ \frac{1}{x^2} \ \left(1-\frac{1}{x}\right)^{\frac{n-1}{2}} \\ f_{\rm WZ}(x) &= -2N \ \frac{\left(1-\frac{1}{x}\right)^{\frac{n+1}{2}}}{x} \\ f(x) &= f_{\rm BI}(x)+f_{\rm WZ}(x) \\ &= \frac{-2N}{x}\left(1-\frac{1}{x}\right)^{\frac{n-1}{2}} \end{aligned} \tag{4.10}$$

Now since $|c|^2 f(|c|^2)$ monotonically increases from 0 to $2N$ as $|c|^2$ varies from 1 to ∞, it is possible to make a (non holomorphic) change of variables to convert the symplectic form ω into to N times the standard Fubini Study form $\omega_{\rm FS}$. Explicitly

$$\omega = -2N\left[\frac{1}{1+|w|^2}\left(\frac{d\bar{w^i}\wedge dw_i}{2i}\right)-\frac{1}{(1+|w|^2)^2}\ \bar{w^i}w_j\ \left(\frac{d\bar{w^j}\wedge dw_i}{2i}\right)\right]. \tag{4.11}$$

where

$$w^i = c^i\sqrt{\frac{\widetilde{f}(|c|^2)}{1-|c|^2\widetilde{f}(|c|^2)}} \tag{4.12}$$

and $\widetilde{f}(x) = f(x)/2N$.

Note that the 'hole' $|c|<1$ has been contracted away to a point in the 'good' w variables (see [7] for a detailed discussion of the same phenomenon in a different context). The quantization of our space is now standard in w variables. The Hilbert space is given by the holomorphic polynomials of $(k+2)$ variables (w_i and 1) with k charge operators $L_i = w_i\partial_{w_i}$. This is identical to the Hilbert space of N identical non-interacting bosons whose single particle Hilbert space consists of $(k+1)$ states with the ith state having charge one under L_i and charge zero under all others and the $(k+2)$th state having all charges zero.

In the rest of this section we present the computations that lead to (4.10).

4.2 Calculation of $\omega_{\rm BI}$

It is convenient to calculate $\omega_{\rm BI}$ in the rotated coordinate system where linear polynomial takes the form of (4.7). As coordinates on the (*unit* $S^{n+2} = \Omega_{n+2}$) we will choose

$$z_{k+1} = Z = \rho e^{i\theta} \tag{4.13}$$

together with $[Z_1, Z_2, \ldots, Z_k, (H)]$ where

$$\frac{[z_1, z_2, \ldots, z_k, (h)]}{\sqrt{1-\rho^2}} \equiv [Z_1, Z_2, \ldots, Z_k, (H)]. \tag{4.14}$$

Note that $H^2 + \sum_{i=1}^k |Z_i|^2 = 1$. Of course the coordinate H simply does not exist - and so may be set to zero in all equations - when n is odd.

Since $\omega_{\rm BI}$ is an exact form it can be expressed as $d\Theta_{\rm bI}$ where $\Theta_{\rm BI}$ is an one-form given by

$$\Theta_{\rm BI} = \int d^n\sigma (P_Z \delta Z + P_{\bar{Z}} \delta \bar{Z}) \tag{4.15}$$

where $Z = z_k$, P_Z is the momentum corresponding to the coordinate Z evaluated at the solution given by (4.7), δZ and $\delta\bar{Z}$ is the fluctuation of Z and $\bar{Z}$ (to the leading order) as the coefficients of the linear polynimial are varied infinitesimally, and we have used the fact that $P_{Z_i} = P_{\bar{Z}_i} = P_H = 0$ on the solution (4.7).

P_Z and $P_{\bar{Z}}$ can be computed from the BI part of the lagrangian which is given by

$$\mathcal{L}_{\rm BI} = -\frac{N}{\Omega_n}\,\sqrt{g^{\Omega_n}}\,\frac{(1-Z\bar{Z})^{\frac{n}{2}}}{R}\sqrt{1 - R^2\,\frac{\bar{Z}^2\dot{Z}^2 + Z^2\dot{\bar{Z}}^2 + 2(2-Z\bar{Z})\dot{Z}\dot{\bar{Z}}}{4(1-Z\bar{Z})}} \tag{4.16}$$

((4.16) may also be rewritten in terms of ρ and θ; see appendix A)

Therefore

$$P_Z = \frac{\partial \mathcal{L}_{\rm BI}}{\partial \dot{Z}} = (i)\frac{N}{\Omega_n}\sqrt{g^{\Omega_n}}(1-Z\bar{Z})^{\frac{n-3}{2}}\left(Z\frac{\bar{Z}^2}{4} - \bar{Z}\left(\frac{2-Z\bar{Z}}{4}\right)\right) \tag{4.17}$$

where we have used $\dot{Z} = \frac{iZ}{R}$ on (4.7). $P_{\bar{Z}}$ is simply the complex conjugate of P_Z.

We will find it useful to compute $\Theta_{\rm BI}$ in terms of the variables c_i. Consider the giant graviton

$$(c_0 + \delta c_0)\, e^{i\frac{t}{R}} z_{k+1} + \sum_{i=1}^{k} \delta a_i e^{i\frac{t}{R}} z^i = 1. \tag{4.18}$$

This configuration is a small fluctuation about (4.7); the corresponding variation in Z is given by

$$\delta Z = -\frac{e^{i\frac{t}{R}}}{c_0^2}\delta c_0 - \frac{\sqrt{1-ZZ}}{c_0}\left(Z_1\delta a_1 + Z_2\delta a_2 + \ldots + Z_k\delta a_k\right); \tag{4.19}$$

$\delta\bar{Z}$ is given by the complex conjugate expression.

Using these expressions

$$\Theta_{\rm BI} = N\left(\frac{i}{2}\right)\frac{1}{|c_0|^4}\,\left(1 - \frac{1}{|c_0|^2}\right)^{\frac{n-1}{2}}\,(\bar{c_o}\delta c_0 - c_0\delta\bar{c_0}) \tag{4.20}$$

(all terms proportional to δa_i evaluate to zero upon integrating over the sphere).

Substituting $|c_0|^2 \to \bar{c^i}c_i = |c|^2$, $\bar{c_0}\delta c_0 \to \bar{c^i}dc_i$ and $c_0\delta\bar{c_0} \to c_i d\bar{c^i}$

$$\Theta_{\rm BI} = N\frac{i}{2}\frac{1}{|c|^4}\,\left(1 - \frac{1}{|c|^2}\right)^{\frac{n-1}{2}}\,\left(\bar{c^i}\delta c_i - c_i\delta\bar{c^i}\right) \tag{4.21}$$

$$\omega_{\rm BI} = d\Theta_{\rm BI} = f_{\rm BI}(|c|^2)\,\left(\frac{d\bar{c^i}\wedge dc_i}{2i}\right) + f'_{\rm BI}(|c|^2)\,\bar{c^i}c_j\,\left(\frac{d\bar{c^j}\wedge dc_i}{2i}\right) \tag{4.22}$$

Where

$$f_{\rm BI}(x) = -2N\,\frac{1}{x^2}\left(1 - \frac{1}{x}\right)^{\frac{n-1}{2}} \tag{4.23}$$

4.3 Calculation of $\omega_{\rm WZ}$

As argued in appendix C.1 of [7] $\omega_{\rm WZ}$ can be written as $\frac{2\pi N}{R^{n+2}\Omega_{n+2}}$ times the volume swept out by the two deformations of the brane surface. The volume of the giant graviton $R^n\Omega_n(1-Z\bar{Z})^{\frac{n}{2}}$; the volume swept out when the graviton is deformed is

$$V_{\rm deformed} = -\int d^n\sigma R^{n+2}\sqrt{g^{\Omega_n}}\ (1-\rho^2)^{\frac{n-1}{2}}\ \rho\delta\rho\wedge\delta\theta \tag{4.24}$$

Here ρ and θ are the radial and angular directions in unit S^{n+2} that are perpendicular to S^n. Using (4.13)

$$\omega_{\rm WZ} = -\frac{2\pi N}{R^{n+2}\Omega_{n+2}}\ V_{\rm deformed} = \frac{2\pi N}{\Omega_{n+2}}\int d^n\sigma\sqrt{g^{\Omega_n}}\ (1-Z\bar{Z})^{\frac{n-1}{2}}\ \left(\frac{\delta\bar{Z}\wedge\delta Z}{2i}\right) \tag{4.25}$$

Using (4.19)

$$\omega_{\rm WZ} = -\frac{2\pi N}{\Omega_{n+2}}\int d^n\sigma\sqrt{g^{\Omega_n}}\ (1-Z\bar{Z})^{\frac{n-1}{2}}\ \left(\frac{\delta\bar{c_0}\wedge\delta c_0}{2i|c_0|^4} + \frac{(1-Z\bar{Z})}{|c_0|^2}\ \sum_{i=1}^{k-1}\left(|Z_i|^2\frac{\delta\bar{a}_i\wedge\delta a_i}{2i}\right)\right) \tag{4.26}$$

Now

$$\int d^n\sigma\sqrt{g^{\Omega_n}}|Z_i|^2\ =\ \frac{2}{n+1}\ \Omega_n. \tag{4.27}$$

$$\delta\bar{c_0}\wedge\delta c_0\ \rightarrow\ \bar{c^i}c_j\ \left(\frac{d\bar{c^j}\wedge dc_i}{|c|^2}\right) \tag{4.28}$$

and

$$\sum_{i=1}^{k-1}(\delta\bar{a}_i\wedge\delta a_i)\rightarrow d\bar{c^i}\wedge dc_i - \bar{c^i}c_j\ \left(\frac{d\bar{c^j}\wedge dc_i}{|c|^2}\right)$$

Using these equations we find

$$\omega_{\rm WZ} = f_{\rm WZ}(|c|^2)\ \left(\frac{d\bar{c^i}\wedge dc_i}{2i}\right) + f'_{\rm WZ}(|c|^2)\ \bar{c^i}c_j\ \left(\frac{d\bar{c^j}\wedge dc_i}{2i}\right) \tag{4.29}$$

Where

$$f_{\rm WZ}(x) = -2\pi N\ \left(\frac{2}{n+1}\right)\ \frac{\Omega_n}{\Omega_{n+2}}\ \frac{\left(1-\frac{1}{x}\right)^{\frac{n+1}{2}}}{x} = -2N\ \frac{\left(1-\frac{1}{x}\right)^{\frac{n+1}{2}}}{x} \tag{4.30}$$

5. Discussion

In this note we have argued that the spectrum of $1/4^{\rm th}$ BPS states of the world volume theory and the $1/8^{\rm th}$ BPS states of the theory on M2 branes is very similar to the $1/8^{\rm th}$ BPS cohomology of $\mathcal{N}$=4 Yang Mills theory. One point of difference is that the M5 and M2 brane supersymmetric spectra consists entirely bosonic states, while the Yang Mills cohomology includes fermions. The reason for this difference can be traced to the fact that

the Euclidean rotation group, SO(d), is a reducible in $d = 4$ but irreducible in $d = 6$ and $d = 3$. The BPS states studied in this note are singlets under SO(6) or SO(3) on the M5 or M2 worldvolume, and so are purely bosonic. $1/8^{\text{th}}$ BPS states in Yang Mills theory are singlets only under one of the two SU(2)s in SO(4); a condition that allows fermions to contribute to this cohomology.

The partition functions Z_5^N and Z_2^N have interesting behavior in the large N limit (see [15] for a longer discussion of the analogous behaviour of Z_3^N in Yang Mills theory). For simplicity let us set $\mu_1 = \mu_2 = \nu$ in and $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \chi$ (in (3.24) and (3.25)) for discussion of this paragraph. At fixed ν and χ, (3.24) and (3.25) simply reduce to the relatively structureless formulas (2.3) and (2.6) in the large N limit. If, however, the large N limit is taken keeping $\widetilde{\nu} = \nu N^{\frac{1}{2}}$ or $\widetilde{\chi} = \chi N^{\frac{1}{4}}$, Z_5^N and Z_2^N undergo sharp Bose Einstein type phase transitions at $\widetilde{\nu}$ and $\widetilde{\chi}$ of order unity. In the large $\widetilde{\nu}$ or $\widetilde{\chi}$ 'Bose condensed' phase we recover the partition functions (2.3) and (2.6). However the small $\widetilde{\nu}$ or $\widetilde{\chi}$ phase has different thermondynamics. It would be interesting to understand the gravity dual of this new phase (see [7] for a related discussion).

The BPS states we have investigated in this note were easily analyzed because they controlled by the relatively tame 'diagonal' dynamics of the Coulomb branch. 1/16 BPS states in these theories are almost certain to explore the non Abelian dynamics of these theories in a detailed way. While the investigation of these states appears to be a rather difficult problem, it holds the promise of substantial pay offs.

Acknowledgments

We would like to thank J. Bhattacharya, N. Dorey, L. Grant, S. Lahiri, S. Kim, G. Mandal and S. Trivedi and all the students in the TIFR theory room for useful discussions. We also thank S. Lahiri and G. Mandal for comments on an advance version of this manuscript. The work of S.M. was supported in part by a Swarnajayanti Fellowship. We would like to thank the University of Hyderabad for hospitality while this paper was being written. We must also acknowledge our debt to the steady and generous support of the people of India for research in basic sciences.

A. Charges of the giant graviton

In this appendix we review the computation of the charges of the giant graviton (4.7) (see for example [5, 19, 18, 16, 20]). Let $z_{k+1} = \rho e^{i\theta}$. The metric on time $\times$ a sphere of radius R may be written as

$$ds^2 = -dt^2 + R^2\left[\frac{d\rho^2}{1-\rho^2} + \rho^2 d\theta^2 + (1-\rho^2)(d\Omega_n)^2\right] \tag{A.1}$$

where $(d\Omega_n)^2$ is the metric on the unit S^n. In this appendix we study the dynamics of a giant graviton that (at a given time) wraps the S^n but is located at some point in ρ and θ. The action for such a giant graviton is given by

$$\mathcal{S} = \int dtL = N\left[-\frac{(1-\rho^2)^{\frac{n}{2}}}{R}\sqrt{1 - R^2\left(\frac{\dot{\rho}^2}{1-\rho^2} + \rho^2\dot{\theta}^2\right)} + (1-\rho^2)^{\frac{n+1}{2}}\dot{\theta}\right] \tag{A.2}$$

$\dot{\theta}$ is constant on all solutions to the equations of motion that follow from (A.2). $\rho =$ *constant* is a solution to the equation of motion provided

$$\dot{\theta} = \frac{1}{R} \tag{A.3}$$

The momentum conjugate to θ on this solution is given by

$$P_\theta = \frac{\partial L}{\partial \dot{\theta}}|_{\dot{\theta}=\frac{1}{R}} = N(1-\rho^2)^{\frac{n-1}{2}} \tag{A.4}$$

The energy of the solution is given by

$$E = \dot{\theta} P_\theta - L = \frac{N}{R}(1-\rho^2)^{\frac{n-1}{2}} \tag{A.5}$$

Therefore for this solution obeys the 'BPS' relation

$$\text{Energy} = \frac{1}{R}(\text{angular momentum}) \tag{A.6}$$

For the case of $AdS^7 \times S^4$ and $AdS^4 \times S^7$ the value of R is $\frac{1}{2}$ and 2 respectively. (A.6) is literally the BPS bound for these cases.

References

[1] N. Seiberg, *Notes on theories with* 16 *supercharges*, *Nucl. Phys.* **67** *(Proc. Suppl.)* (1998) 158 [hep-th/9705117].

[2] O. Aharony, M. Berkooz and N. Seiberg, *Light-cone description of* $(2,0)$ *superconformal theories in six dimensions*, *Adv. Theor. Math. Phys.* **2** (1998) 119 [hep-th/9712117].

[3] A. Basu and G. Mandal, *Dual giant gravitons in* $AdS(m) \times Y^n$ *(Sasaki-Einstein)*, *JHEP* **07** (2007) 014 [hep-th/0608093].

[4] A. Mikhailov, *Giant gravitons from holomorphic surfaces*, *JHEP* **11** (2000) 027 [hep-th/0010206].

[5] J. McGreevy, L. Susskind and N. Toumbas, *Invasion of the giant gravitons from anti-de Sitter space*, *JHEP* **06** (2000) 008 [hep-th/0003075].

[6] C.E. Beasley, *BPS branes from baryons*, *JHEP* **11** (2002) 015 [hep-th/0207125].

[7] I. Biswas, D. Gaiotto, S. Lahiri and S. Minwalla, *Supersymmetric states of* $N = 4$ *Yang-Mills from giant gravitons*, hep-th/0606087.

[8] G. Mandal and N.V. Suryanarayana, *Counting* 1/8*-BPS dual-giants*, *JHEP* **03** (2007) 031 [hep-th/0606088].

[9] S. Minwalla, *Restrictions imposed by superconformal invariance on quantum field theories*, *Adv. Theor. Math. Phys.* **2** (1998) 781 [hep-th/9712074].

[10] M. Günaydin, P. van Nieuwenhuizen and N.P. Warner, *General construction of the unitary representations of anti-de Sitter superalgebras and the spectrum of the* S^4 *compactification of eleven-dimensional supergravity*, *Nucl. Phys.* **B 255** (1985) 63.

JHEP12(2007)004

[11] S. Minwalla, *Particles on AdS(4/7) and primary operators on $M(2/5)$ brane worldvolumes*, *JHEP* **10** (1998) 002 [hep-th/9803053].

[12] O. Aharony, Y. Oz and Z. Yin, *M-theory on $AdS(p) \times S(11-p)$ and superconformal field theories*, *Phys. Lett.* **B 430** (1998) 87 [hep-th/9803051].

[13] A. Casher, F. Englert, H. Nicolai and M. Rooman, *The mass spectrum of supergravity on the round seven sphere*, *Nucl. Phys.* **B 243** (1984) 173.

[14] M. Günaydin and N.P. Warner, *Unitary supermultiplets of* $\mathrm{Osp}(8/4, R)$ *and the spectrum of the S_7 compactification of eleven-dimensional supergravity*, *Nucl. Phys.* **B 272** (1986) 99.

[15] J. Kinney, J.M. Maldacena, S. Minwalla and S. Raju, *An index for 4 dimensional super conformal theories*, *Commun. Math. Phys.* **275** (2007) 209 [hep-th/0510251].

[16] A. Hashimoto, S. Hirano and N. Itzhaki, *Large branes in AdS and their field theory dual*, *JHEP* **08** (2000) 051 [hep-th/0008016].

[17] D. Berenstein, *Large-N BPS states and emergent quantum gravity*, *JHEP* **01** (2006) 125 [hep-th/0507203].

[18] M.T. Grisaru, R.C. Myers and O. Tafjord, *SUSY and Goliath*, *JHEP* **08** (2000) 040 [hep-th/0008015].

[19] S.R. Das, A. Jevicki and S.D. Mathur, *Vibration modes of giant gravitons*, *Phys. Rev.* **D 63** (2001) 024013 [hep-th/0009019].

[20] P. Ouyang, *Holomorphic D7-branes and flavored $N = 1$ gauge theories*, *Nucl. Phys.* **B 699** (2004) 207 [hep-th/0311084].

JHEP12(2007)004